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Soliton-like solutions of a generalized variable-coefficient higher order nonlinear Schrödinger equation from inhomogeneous optical fibers with symbolic computation

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Abstract

For the long-distance communication and manufacturing problems in optical fibers, the propagation of subpicosecond or femtosecond optical pulses can be governed by the variable-coefficient nonlinear Schrödinger equation with higher order effects, such as the third-order dispersion, self-steepening and self-frequency shift. In this paper, we firstly determine the general conditions for this equation to be integrable by employing the Painlevé analysis. Based on the obtained 3×3 Lax pair, we construct the Darboux transformation for such a model under the corresponding constraints, and then derive the nth-iterated potential transformation formula by the iterative process of Darboux transformation. Through the one- and two-soliton-like solutions, we graphically discuss the features of femtosecond solitons in inhomogeneous optical fibers.

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1. Introduction

The nonlinear Schrödinger (NLS) equation, as an important physical model, describes the dynamics of optical soliton propagation in nonlinear optical fibers [1–11]. Since the optical soliton in a dielectric fiber was theoretically and experimentally discovered [12], it has been regarded as the natural data bits and as an important alternative to the next generation of ultrafast optical telecommunication systems [13–18]. In realistic optical fibers, due to the

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long-distance communication and manufacturing problems, the propagation of subpicosecond or femtosecond optical pulses is governed by the variable-coefficient higher order NLS (HNLS) equations [4–9, 19, 20], which include various higher order effects influenced by the spatial variations of the fiber parameters such as the third-order dispersion (TOD), self-steepening and self-frequency shift [1–5]. Furthermore, many properties of these HNLS models have been analyzed by some authors from different points of view [1–6].

In reality, the fundamental soliton propagation cannot exist in standard fiber (which is of the order of 0.2 dB km^{-1} at carrier wavelength 1.55 μm), but the stable pulse propagation over a considerable distance can still be obtained by appropriate combination of dispersion and optical amplification [21]. The variable-coefficient NLS models have attracted a great deal of interest in dispersion-managed optical fibers [4–8, 22, 23]. For the subpicosecond or femtosecond optical soliton control, the above higher order effects like TOD, self-steepening and self-frequency shift should be taken into account. In view of the inhomogeneous fibers, the problem of femtosecond soliton control can be governed by the generalized variable-coefficient HNLS equation [4–6]

$$\begin{aligned} &\mathrm{i}\, u_z + a(z)\, u_{tt} + b(z)\, |u|^2\, u + \mathrm{i}\, c(z)\, u_{ttt} + \mathrm{i}\, d(z)\, (|u|^2 u)_t + \mathrm{i}\, e(z)\, u\, (|u|^2)_t \\ &\qquad + \mathrm{i}\, f(z)\, u_t + [g(z) + \mathrm{i}\, h(z)]\, u = 0, \end{aligned} \tag{1}$$

where $u(z, t)$ is the complex envelope of the electrical field in the comoving frame, z and t, respectively, represent the normalized propagation distance along the fiber and retarded time, while all the variable coefficients are real analytic functions. $a(z)$ and $c(z)$ denote the group velocity dispersion (GVD) and TOD, respectively. $b(z)$ accounts for the self-phase modulation (SPM), while $d(z)$ is the self-steepening (also called the Kerr dispersion) and $e(z)$ is related to the delayed nonlinear response effects. The term proportional to $f(z)$ results from the group velocity and $h(z)$ represents the amplification or absorption coefficient. Equation (1) can also be extensively used to describe the telecommunication and ultrafast signal-routing systems in the weakly dispersive and nonlinear dielectrics with distributed parameters [4–6].

When $c(z) = d(z) = e(z) = 0$, equation (1) degenerates to [9–11]

$$\mathrm{i}\, u_z + a(z)\, u_{tt} + b(z)\, |u|^2\, u + \mathrm{i}\, f(z)\, u_t + [g(z) + \mathrm{i}\, h(z)]\, u = 0, \tag{2}$$

in which the optical soliton can be formed based on the exact balance between the GVD and SPM effects. Some analytic soliton-like solutions for equation (2) have been obtained and the relevant properties have also been discussed in detail [9–11]. If $c(z) = e(z) = 0$, equation (1) reduces to the variable-coefficient derivative NLS equation [24], which describes the optical soliton propagation in the presence of Kerr dispersion. However, to model the effects of pulse broadening in some particular regions as the frequency region, one should consider the TOD effect and ignore the GVD effect [1]. In this case, the asymmetrical broadening between these higher order effects can balance themselves to achieve the soliton pulse propagation in fiber systems [1–6]. When $f(z) = 0$ and/or $g(z) = 0$, the analytic multi-soliton-like solutions of equation (1) have been given under a special condition for this equation to be Painlevé integrable [4–6]. Nevertheless, as shown in [25–27], there may exist another constraint for such a variable-coefficient nonlinear evolution equation (NLEE) to admit soliton solutions.

With symbolic computation [27–30], in section 2, we will firstly determine the general conditions for equation (1) to be integrable by employing the Painlevé analysis. It will be found that there exist two kinds of constraints for equation (1) to possess the soliton solutions, one of which is consistent with the condition presented in [5, 6]. Thus, we will devote ourselves to studying equation (1) under another set of constraints and then generalize the 2×2 Lax pair to the 3×3 linear eigenvalue problem. In section 3, we will construct the Darboux transformation based on the obtained 3×3 Lax pair, and derive the nth-iterated potential transformation formula by iterating the Darboux transformation n times. Section 4 will present the graphical discussions about the features of solitons propagating in inhomogeneous optical fibers through the one- and two-soliton-like solutions of equation (1). Our conclusions will be addressed in section 5.

2. Painlevé analysis and Lax pair for equation (1) with symbolic computation

In this section, to determine the necessary conditions for equation (1) to be completely integrable, we will employ the Weiss–Tabor–Carnevale (WTC) method [31] and the simplified Kruskal ansatz [32] to carry out the Painlevé analysis.

According to the WTC method, if the solutions of a given partial differential equation (PDE) are 'single-valued' about the movable singularity manifolds, then this PDE has the Painlevé property. In order to carry out the Painlevé analysis, we introduce $v(z,t) = u^*(z,t)$, where * represents the complex conjugate. Then, equation (1) turns out to be the following set of equations:

$$\begin{aligned} \mathrm{i}\, u_z + a(z)\, u_{tt} + b(z)\, u^2\, v + \mathrm{i}\, c(z)\, u_{ttt} + \mathrm{i}\, d(z)\, (u^2 v)_t + \mathrm{i}\, e(z)\, u\, (uv)_t \\ + \mathrm{i}\, f(z)\, u_t + [g(z) + \mathrm{i}\, h(z)]\, u = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} \mathrm{i}\, v_z - a(z)\, v_{tt} - b(z)\, v^2 u + \mathrm{i}\, c(z)\, v_{ttt} + \mathrm{i}\, d(z)\, (v^2 u)_t + \mathrm{i}\, e(z)\, v\, (uv)_t \\ + \mathrm{i}\, f(z)\, v_t - [g(z) - \mathrm{i}\, h(z)]\, v = 0. \end{aligned} \tag{4}$$

The generalized Laurent series expansions of u and v are of the form

$$u = \phi^p \sum_{j=0}^{\infty} u_j(z,t)\phi^j, \tag{5}$$

$$v = \phi^q \sum_{j=0}^{\infty} v_j(z,t)\phi^j, \tag{6}$$

where p and q are two negative integers, u_j, v_j and $\phi(z,t)$ are all analytic functions of z and t in a neighborhood of the noncharacteristic singular manifold $\phi(z,t) = t + \psi(z) = 0$, with $\psi(z)$ as an arbitrary analytic function of z. Through the leading order analysis, we obtain $p = q = -1$ and $u_0 v_0 = -6c(z)/[3d(z) + 2e(z)]$. With symbolic computation, it is found that there are two cases for equations (3) and (4) to pass the Painlevé test as follows: (i) $d(z) = -e(z)$; (ii) $d(z) = -2\, e(z)$.

For case (i), the resonances occur at $j = -1, 0, 1, 3, 4, 5$, of which $j = -1$ corresponds to the arbitrariness of the singular manifold and $j = 0$ shows that either u_0 or v_0 is arbitrary, while the compatibility conditions at $j = 1, 3, 4, 5$ are satisfied identically, if the variable coefficients obey the following constraints:

$$a(z) = -3\frac{b(z)c(z)}{e(z)}, \qquad e(z) = \alpha c(z)\, \mathrm{e}^{2\int h(z)\, \mathrm{d}z}, \tag{7}$$

where α is an arbitrary constant, which are identical with the conditions given in [4–6]. In this case, equation (1) becomes the Hirota equation with variable coefficients and has been investigated by some authors [6].

For case (ii), the resonances occur at $j = -1, 0, 2, 3, 4, 4$, of which $j = -1$ corresponds to the arbitrariness of the singular manifold and $j = 0$ shows that either u_0 or v_0 is arbitrary. With the aid of symbolic computation, we can also prove that the compatibility conditions at $j = 2, 3, 4, 4$ are satisfied automatically, if the variable coefficients obey the following constraints:

$$a(z) = -\frac{3\sigma c(z)}{2\delta}, \qquad e(z) = \frac{\delta b(z)}{\sigma} = \delta\, e^{2\int h(z)\, dz} c(z), \tag{8}$$

where $\delta \neq 0$ and $\sigma \neq 0$ are a couple of arbitrary constants. When $f(z) = g(z) = h(z) = 0$ and other variable coefficients are all constants, equation (1) with constraints (8) reduces to the constant-coefficient HNLS equation [33].

Therefore, under constraints (7) or (8), we can say that equation (1) possesses the Painlevé property, and either of these two constraints has nothing to do with the variable coefficients $f(z)$ and $g(z)$. It is noted that the special constraint conditions shown in [4–6, 8] are identical with constraints (7), under which some properties such as the multi-soliton solutions have been discussed by the Darboux transformation and Hirota method. To our knowledge, the investigation of equation (1) under constraints (8) has not been widespread.

In soliton theory, the Lax pair is of fundamental importance in that it not only gives a scheme to solve the initial problem of a given NLEE through the method of inverse scattering, but also plays a vital role in studying the integrable properties of NLEEs such as the Hamiltonian structures, conservation laws, symmetry classes and Darboux transformations [30, 34–40]. In the following research, we will stay with the second set of constraints and employ the Ablowitz–Kaup–Newell–Segur (AKNS) procedure [36] to construct the Lax pair for equation (1). It is noted that the key for constructing a Lax pair for a given NLEE through the AKNS method is to choose an appropriate linear eigenvalue problem [30, 35–37]. According to the constraints on those variable coefficients, we generalize the 2×2 Lax pair for equation (1) under constraints (8) to the 3×3 linear eigenvalue problem. Without loss of generality, we assume that $\sigma = 1$ and $\delta = -1$. Thus, the linear eigenvalue problem for equation (1) under constraints (8) can be expressed as follows:

$$\Phi_t = \mathbf{U}\Phi = (\lambda U_0 + U_1)\Phi, \qquad \Phi_z = \mathbf{V}\Phi = (\lambda^3 V_0 + \lambda^2 V_1 + \lambda V_2 + V_3)\Phi, \tag{9}$$

where $\Phi = (\phi_1, \phi_2, \phi_3)^T$, T denotes the transpose of the matrix and λ is a spectral parameter, while the matrices U_0, U_1, V_0, V_1, V_2 and V_3 are presented in the forms

$$U_0 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & ku & k^*u^* \\ -k^*u^* & 0 & 0 \\ -ku & 0 & 0 \end{pmatrix}, \quad V_0 = 4c(z)U_0, \quad V_1 = 4c(z)U_1, \tag{10}$$

$$V_2 = \begin{pmatrix} A_1 + \frac{2i}{3}c(z)\, e^{2\int h(z) dz}|u|^2 & kA_2 & -k^*A_2^* \\ -k^*A_2^* & A_1^* & -k^{*2}A_3^* \\ kA_2 & k^2A_3 & A_1^* \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 0 & kA_4 & k^*A_4^* \\ -k^*A_4^* & A_5 & 0 \\ -kA_4 & 0 & A_5^* \end{pmatrix}, \tag{11}$$

with

$$k = \frac{1}{\sqrt{3}}\, e^{-\frac{i}{4}\left\{2t - \int [c(z) - 4g(z) + 2f(z)] dz\right\} + \int h(z)\, dz},$$

$$A_1 = i f(z) + ic(z)\left(\frac{3}{4} + \frac{2}{3}\, e^{2\int h(z)\, dz}|u|^2\right),$$

$$A_2 = c(z)\,(u + 2\, iu_t)\,,$$

$$A_3 = -2\, ic(z)u^2,$$

$$A_4 = c(z)\left[-\frac{4}{3}\,\mathrm{e}^{2\int h(z)\,\mathrm{d}z}|u|^2u - u_{tt} + \mathrm{i}u_t - \frac{1}{2}u\right] - f(z)u,$$

$$A_5 = \frac{1}{3}\,\mathrm{e}^{2\int h(z)\,\mathrm{d}z}c(z)(u^*u_t - uu_t^* - \mathrm{i}|u|^2).$$

With constraints (8), it is easy to prove that the compatibility condition $\mathbf{U}_z - \mathbf{V}_t + [\mathbf{U}, \mathbf{V}] = 0$ gives rise to equation (1).

In order to reveal the analytic soliton-like solutions for equation (1) under constraints (8), we will employ the Darboux transformation method, which is an effective and computerizable procedure and has been widely used to construct soliton-like solutions for a class of variable-coefficient NLEEs [5–7]. The Darboux transformation can give rise to a general procedure to recursively generate a series of analytic solutions including the multi-soliton solutions from an initial solution [34–40]. It is shown that an obvious advantage of the Darboux transformation lies in that the iterative algorithm is purely algebraic and very computerizable by virtue of symbolic computation [34–40].

3. Darboux transformation with symbolic computation

In this section, on the basis of the 3×3 Lax pair and under constraints (8), we construct the Darboux transformation for equation (1) as the following form [37]:

$$\Phi' = D\Phi = (\lambda I + S)\Phi, \tag{12}$$

where I is the 3×3 identity matrix, S is a nonsingular matrix and its entries $s_{ij}\,(1 \leqslant i, j \leqslant 3)$ are all parameters to be determined. It requires that Φ' should also satisfy the linear eigenvalue problem (9), i.e.,

$$\Phi'_t = (\lambda U'_0 + U'_1)\Phi', \qquad \Phi'_z = (\lambda^3 V'_0 + \lambda^2 V'_1 + \lambda V'_2 + V'_3)\Phi', \tag{13}$$

where U'_0, U'_1, V'_0, V'_1, V'_2 and V'_3 have the same forms as U_0, U_1, V_0, V_1, V_2 and V_3 except that $u(z, t)$ is replaced by $u_1(z, t)$. It is noted that the key point in the Darboux transformation is to make the linear eigenvalue problem (9) invariant under transformation (12). Thus, using the knowledge of the Darboux transformation [37], we can obtain the following equations:

$$U'_0 = U_0, \qquad V'_0 = V_0, \tag{14}$$

$$U'_1 - U_1 + U_0S - SU_0 = 0, \tag{15}$$

$$U'_1S - SU_1 - S_t = 0, \tag{16}$$

$$V'_1 - V_1 + V_0S - SV_0 = 0, \tag{17}$$

$$V'_2 - V_2 + V'_1S - SV_1 = 0, \tag{18}$$

$$V'_3 - V_3 + V'_2S - SV_2 = 0, \tag{19}$$

$$V'_3S - SV_3 - S_z = 0, \tag{20}$$

where equations (15) and (17) are actually identical and are satisfied if and only if

$$s_{12} = \frac{\mathrm{i}k}{2}(u - u_1), \tag{21}$$

$$s_{31} = s_{12}, \qquad s_{21} = s_{13} = -s_{12}^*. \tag{22}$$

Then, based on the investigation in [38–40], we can specially define

$$S = -H\Lambda H^{-1}, \tag{23}$$

with

$$H = \begin{pmatrix} \phi_{11}(\lambda_1) & \phi_{21}(\lambda_1) & \phi_{21}(\lambda_1) \\ \phi_{21}(\lambda_1) & -\phi_{11}(\lambda_1) & 0 \\ \phi_{31}(\lambda_1) & 0 & -\phi_{11}(\lambda_1) \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}, \tag{24}$$

where $Re(\lambda_1) = 0$, $\phi_{31}(\lambda_1) = \phi_{21}(\lambda_1)$ and the first column of H is the real vector solution of the Lax pair for the initial potential $u(z, t)$ with $\lambda = \lambda_1$. Owing to the strict constraints on the entries of S matrix, we can find that the construction of the Darboux transformation for the 3×3 Lax pair is different from that in [5, 6, 40].

It is easy to verify that expressions (22) are satisfied automatically. From expression (21), we can get the relation between the new potential $u_1(z, t)$ and old potential $u(z, t)$ as below:

$$u_1 = u + \frac{4}{k} \frac{Im\,(\lambda_1)\,\phi_{11}(\lambda_1)\phi_{21}(\lambda_1)}{\phi_{11}^2(\lambda_1) + 2\phi_{21}^2(\lambda_1)}. \tag{25}$$

With the help of symbolic computation, the identity of equations (16) and (18)–(20) can also be proved. Thus, we have constructed the Darboux transformation for equation (1) under constraints (8) and obtained the relationship between two potentials. Analogous to this procedure and iterating the Darboux transformation n times, we find the following nth-iterated potential transformation formula:

$$u_n = u + \frac{4}{k} \sum_{j=1}^{n} \frac{Im(\lambda_j)\phi_{1,j}(\lambda_j)\phi_{2,j}(\lambda_j)}{A_j}, \tag{26}$$

where

$$\phi_{m,j+1}(\lambda_{j+1}) = (\lambda_{j+1} - \lambda_j^*)\phi_{m,j}(\lambda_{j+1}) - \frac{B_j}{A_j}(\lambda_j - \lambda_j^*)\phi_{m,j}(\lambda_j), \tag{27}$$

$$A_j = \phi_{1,j}^2(\lambda_j) + 2\phi_{2,j}^2(\lambda_j), \tag{28}$$

$$B_j = \phi_{1,j}(\lambda_j)\phi_{1,j}(\lambda_{j+1}) + 2\phi_{2,j}(\lambda_j)\phi_{2,j}(\lambda_{j+1}), \tag{29}$$

with $m = 1, 2, 3$, $\phi_{3,j}(\lambda_j) = \phi_{2,j}(\lambda_j)$ and $[\phi_{1,j}(\lambda_j), \phi_{2,j}(\lambda_j), \phi_{3,j}(\lambda_j)]^T$ as the real vector solution of the Lax pair corresponding to λ_j for potential $u_{j-1}(z, t)$ $(j = 1, 2, 3, \ldots, n)$. Obviously, it can be seen that expression (26) provides us with a general procedure to generate the multi-soliton-like solutions of equation (1).

4. Soliton-like solutions of equation (1) and applications in inhomogeneous fibers

From what have been obtained in section 3, we know that some soliton-like solutions for equation (1) under constraints (8) can be derived by solving the linear eigenvalue problem (9) with an initial potential and performing tedious but not complicated algebraic operations [38–40]. Substituting the zero seed solution of equation (1) into system (9), we can get the one-soliton-like solution from expression (25) as below:

$$u = \sqrt{6}\mu_1\, \mathrm{e}^{\frac{1}{4}\mathrm{i}\{2t - \int[c(z)+2f(z)-4g(z)]\mathrm{d}z + 4\mathrm{i}\int h(z)\mathrm{d}z\}} \times \mathrm{sech}\left\{2\mu_1 t - \frac{\mu_1}{2}\int \left[16\mu_1^2 c(z) + 3c(z) + 4f(z)\right]\mathrm{d}z\right\}, \tag{30}$$

with $\lambda_1 = \mathrm{i}\mu_1$ and μ_1 is an arbitrary real constant. It is shown in solution (30) that the pulse width mainly depends on μ_1 and the velocity of the femtosecond soliton is determined by $\frac{\mu_1}{2}\int\left[16\mu_1^2 c(z) + 3c(z) + 4f(z)\right]\mathrm{d}z$. The soliton amplitude $|\sqrt{6}\mu_1\, \mathrm{e}^{-\int h(z)\,\mathrm{d}z}|$ is not only

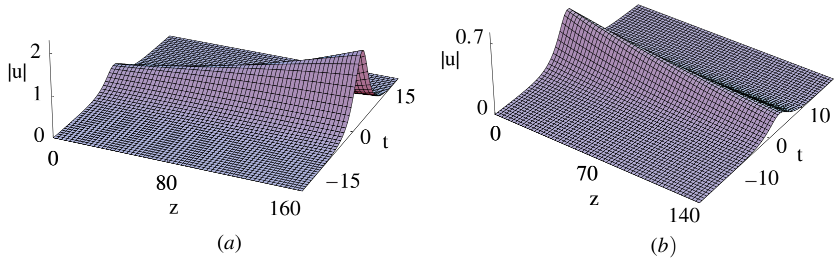


Figure 1. The evolution plot of an optical soliton given by solution (30) with $\mu_1 = 0.2$, $c(z) = 0.02$ and $f(z) = g(z) = 0$. (*a*) $p = 0.015$; (*b*) $p = -0.015$.

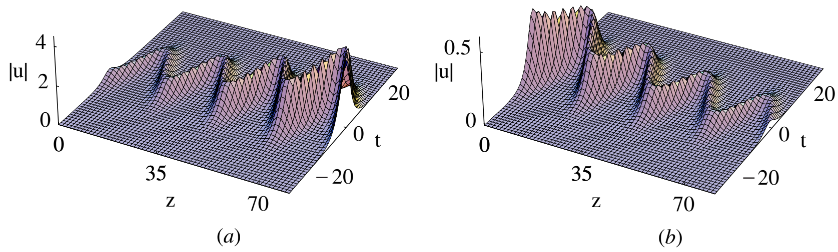


Figure 2. The evolution plot of an optical soliton given by solution (30) with periodic influence for $\mu_1 = 0.2$, $c(z) = 0.02$, $f(z) = 2\sin(0.3z)$ and $g(z) = 0$. (*a*) $p = 0.05$; (*b*) $p = -0.05$.

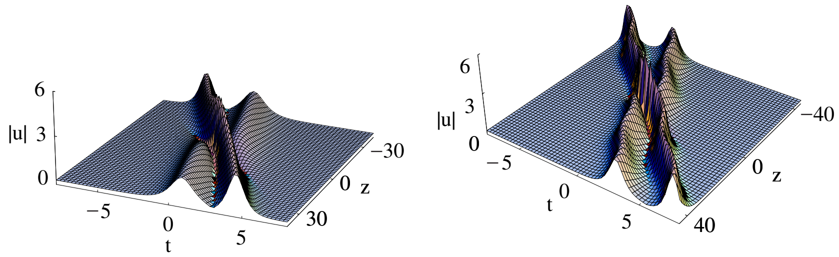
dependent on the parameter μ_1, but also related to the amplification or absorption coefficient $h(z)$. Therefore, through controlling the distributed parameters $c(z)$, $f(z)$ and $h(z)$, we can discuss some transmission properties of the femtosecond optical solitons in inhomogeneous fiber systems.

When $h(z) = 0$, it can be seen that soliton amplitude is a constant. As studied in the realistic optical systems, the amplification or absorption effect often cannot be neglected. Based on the realization of the decreasing GVD in a fiber [22], we can choose the GVD, nonlinearity and TOD parameters according to the results in [5, 7]. Then, the gain/loss function is $h(z) = -p/2$, which represents the dispersion increasing (decreasing) fiber media for $p > 0$ ($p < 0$). In figure 1, we can catch that the soliton amplitude exponentially grows (attenuates) with the velocity $v = \mathrm{e}^{0.0075z}$ ($v = \mathrm{e}^{-0.0075z}$) because of the influence of the amplification (absorption) coefficient. In this case, the soliton velocity and pulse width are both invariant in the soliton propagation along the fiber. Under the periodic influence of $f(z)$, figure 2 provides us with the evolution plot of Solution (30) for different signs of the parameter p, from which we can clearly see that the soliton group velocity is ceaselessly changing along the optical soliton propagation in the fiber.

With symbolic calculations, we can get the energy conservation law of equation (1) under constraints (8) as below:

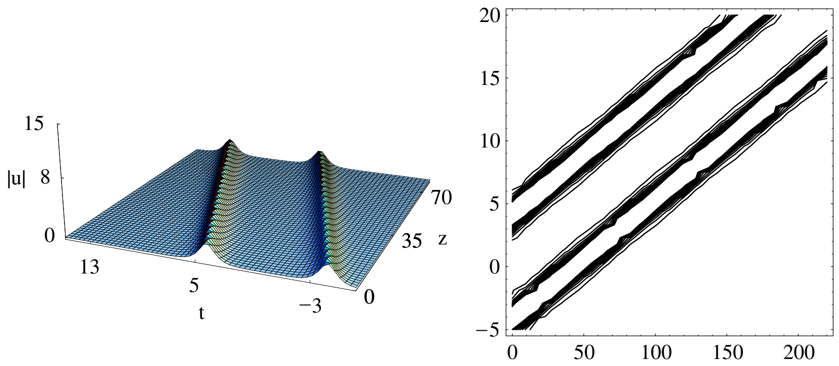
$$
\begin{aligned}
&\mathrm{i}\frac{\partial\left(\mathrm{e}^{2\int h(z)\mathrm{d}z}|u|^2\right)}{\partial z} + \frac{\partial\left(\mathrm{e}^{2\int h(z)\mathrm{d}z}\Pi\right)}{\partial t} = 0,\\
&\Pi = c(z)\left[\mathrm{i}|u|^2_{tt} + 2\,\mathrm{i}\,\mathrm{e}^{2\int h(z)\mathrm{d}z}|u|^4 + \frac{3}{2}(u_t u^* - u u_t^*) - 3\mathrm{i}u_t u_t^*\right] + \mathrm{i}f(z)|u|^2,
\end{aligned}
\tag{31}
$$

where $\mathrm{e}^{2\int h(z)\mathrm{d}z}|u|^2$ and $\mathrm{e}^{2\int h(z)\mathrm{d}z}\Pi$, respectively, represent the conserved density and flux with a modification by multiplication of $\mathrm{e}^{2\int h(z)\mathrm{d}z}$ to counteract the attenuation/growth caused by the linear fiber loss/gain. If $h(z) = 0$, the soliton-like solution is stable and its physical



(*a*) (*b*)

Figure 3. The evolution plot of the amplitudes of two femtosecond solitons via solution (33) with $\mu_2 = 0.1$ and $f(z) = g(z) = h(z) = 0$. (*a*) $\mu_1 = 0.7$ and $c(z) = 0.02$; (*b*) $\mu_1 = 1.3$ and $c(z) = 0.02 + 0.005\sin(0.25z)$.



(*a*) (*b*)

Figure 4. (*a*) The evolution plot of the amplitudes of two femtosecond solitons via solution (33) with $\mu_1 = -0.999$, $\mu_2 = 0.1$, $c(z) = 0.02$ and $f(z) = g(z) = h(z) = 0$; (*b*) the corresponding contour plot.

quantity is conserved, while with the inclusion of the linear loss/gain term $h(z)$ the soliton-like solution is a stationary localized object, which can be used to explain the features of the femtosecond optical soliton demonstrated in figures 1 and 2. Using the vanishing boundary condition for solution (30), it is clear that

$$\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \mathrm{e}^{2\int h(z)\,\mathrm{d}z}|u|^2\mathrm{d}t = 0, \tag{32}$$

which indicates that the quantity $\int_{-\infty}^{\infty}|u|^2\,\mathrm{d}t$ will exponentially decay/grow as the rate $\mathrm{e}^{-2\int h(z)\mathrm{d}z}$.

When $n = 2$, from expression (26), we can obtain the two-soliton-like solution for equation (1) as follows:

$$u(z,t) = \sqrt{2}\,\mathrm{e}^{\frac{1}{4}\mathrm{i}\{2t-\int[c(z)+2f(z)-4g(z)]\mathrm{d}z+4\mathrm{i}\int h(z)\mathrm{d}z\}}$$
$$\times\frac{[\mu_1\cosh(2\varphi_2)-\mu_2\cosh(2\varphi_1)]\left(\mu_1^2-\mu_2^2\right)}{-2\mu_1\mu_2[\sinh(2\varphi_1)\sinh(2\varphi_2)+1]+\cosh(2\varphi_1)\cosh(2\varphi_2)\left(\mu_1^2+\mu_2^2\right)}, \tag{33}$$

where $\lambda_j = \mathrm{i}\mu_j$ and $\varphi_j = \mu_j t - \frac{\mu_j}{4}\left\{\int\left[16\mu_j^2 c(z) + 3c(z) + 4f(z)\right]\mathrm{d}z\right\}(j = 1, 2)$, from which we can see that the pulse width of each soliton in the two-soliton-like solution is determined by the imaginary part μ_j of spectral parameter λ_j. This is different from the previous result in [5, 6]. It is shown that the group velocity of each soliton is related to $\frac{\mu_j}{4}\left\{\int\left[16\mu_j^2 c(z)+3c(z)+4f(z)\right]\mathrm{d}z\right\}$, which means that we can obtain abundant femtosecond soliton structures through appropriately adjusting the distributed parameters in the femtosecond soliton control system. When choosing the TOD coefficient $c(z)$ with different values, figure 3 shows the elastic interactions between two femtosecond optical solitons.

In principle, it is impossible to obtain the separating evolution behavior of solitons for equation (1) under constraints (8) because the real part of the spectral parameter is zero [5]. However, if the ratio $|\mu_1|/|\mu_2| \sim 1$, an approximative separation between two solitons is likely to occur. From figure 4, we can catch that the separation between two solitons approximately keeps constant along two solitons propagating in the optical fiber, which can also be clearly found in the corresponding contour plot.

5. Conclusions

In the real inhomogeneous fiber, the dynamics of the femtosecond soliton propagation are governed by the variable-coefficient HNLS equations with higher order effects such as the TOD, self-steepening and self-frequency shift. Different from previous papers, we firstly present two kinds of constraints for equation (1) to be integrable. It is worth noting that either of these two constraints has nothing to do with the variable coefficients $f(z)$ and $g(z)$. For the former, i.e., constraints (7), some integrable properties of equation (1) have been studied by some authors in recent literature. Due to the strict constraint conditions among the parameters GVD, SPM, TOD, self-steepening and self-frequency shift, it has been shown that the constructions of 3×3 Lax pair and Darboux transformation for equation (1) with constraints (8) are distinct from those in previous papers. In this paper, we have devoted ourselves to equation (1) with constraints (8), under which the multi-soliton-like solutions for equation (1) have been obtained by employing the Darboux transformation based on the 3×3 Lax pair. Though controlling the GVD, TOD, SPM, self-steepening and amplification (absorption) parameters in the femtosecond soliton control systems, we have discussed some potential applications in the inhomogeneous optical fiber systems by the one- and two-soliton-like solutions.

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